FAMILIES OF CENTRAL ORBITS RELATED TO CIRCULAR TRAJECTORIES.*

By FRANK LOXLEY GRIFFIN, Williams College.

§ 1. CIRCLES AS CENTRAL ORBITS.

1. THE PROBLEM. Given any central force which is everywhere finite and real, and which is a function of the distance alone, $f=f(r)$, it is known that all circles with centers at the center of force are possible orbits. For if a particle were to move in such a circle, of radius a, the centrifugal force v^2/a would have to balance exactly the attraction $f(a)$; and, conversely, if the particle were started at a distance a from the center of force, in a direction perpendicular to the radius vector, and with the velocity defined by $v^2=af(a)$, the circular orbit would result.

But in order that a circle whose center is elsewhere than at the center of force, be described as a central orbit, the force must vary according to a *special* law, which is obtainable when the polar equation of the given orbit has been written. The object of this paper is (1) to derive such a law of force, (2) to ascertain what other circular orbits are admitted by it and what properties have they in common, (3) to learn what other laws admit families of orbits having some of those properties, and (4) to point out the families in question in the case of two simple laws.

2. LAWS ADMITTING CIRCULAR ORBITS. Let the center of force be selected as the pole, and let the polar angle θ be measured from the line through the pole and the center of the circle; then the equation of the orbit is

(1) $$2d\cos \theta = 1/u - (a^2 - d^2)u,$$

where u is the reciprocal of the radius vector, a is the radius of the circle, and d is the distance between the pole and center. [It is necessary to take

$d<a$, since a closed central orbit not containing the center of force is impossible.*]

The required central force must vary along the orbit according to the law: $f=h^2u^2(u+d^2u/d\,\theta^2)$, where h is the constant of areas. The differentiation of (1) with respect to θ, followed by the elimination of θ from the two equations gives

$$(2) \qquad 4d^2=[1/u^2-(a^2-d^2)]^2u^2+[1/u^2+(a^2-d^2)]^2(du/d\,\theta)^2,$$

whence is obtained by adding $4(a^2-d^2)$ to both members:

$$(3) \qquad u^2+(du/d\,\theta)^2=4a^2u^4[(a^2-d^2)u^2+1]^{-2}.$$

By differentiating again, or by noting that the curvature, given by $u^3(u+d^2u/d\,\theta^2)[u^2+(du/d\,\theta)^2]^{-\frac{3}{2}}$, equals $1/a$ at all points of the circle, the necessary force is easily obtained as a function of u alone:

$$(4) \qquad f=\frac{m^2u^5}{(n^2\,u^2+1)^3}=\frac{m^2r}{(n^2+r^2)^3},$$

where $m^2=8a^2h^2$ and $n^2=a^2-d^2$. Some obvious properties of this force are: (a) it vanishes both "at infinity" and at the center of force; (b) it is everywhere real; and (c) it is everywhere finite and continuous. If the values of a, d, and h had been different in the given orbit, the values of m and n would usually be different; but the properties mentioned would persist. The type of law where the force is given by (4) for particular values of m and n will be called Type I hereafter.

§ 2. OTHER CIRCULAR ORBITS FOR LAWS OF TYPE I.

3. A FAMILY OF CIRCLES. Consider any particular law of Type I. Because of its properties, just enumerated, this law admits the infinitude of circular orbits whose centers are at the pole. But, leaving aside this somewhat trivial family, the question arises as to the possibility of further circular orbits besides the original one. Evidently in all such orbits which may be admitted by the given law, a^2h^2 and a^2-d^2 must be invariant, for if either m or n changes value, the law changes.

The constancy of the second expression restricts the set of circles admissible as orbits; that of the first imposes a restriction upon the initial conditions. The totality of possible circles is represented by the equation

$$(5) \qquad 2D\cos(\theta-K)=1/u-n^2u,$$

---

*Cf. E. J. Routh, *Dynamics of a Particle*, pp. 292-3.

obtained from (2) by replacing the original d and a by D and A the corresponding constants for any other circle, imposing the restriction $A^2 - D^2 = n^2$, and introducing K as the longitude of the center of the circle. There is a double infinitude of these circles, obtained by varying D and K.

Moreover, as will now be shown, there exist real initial conditions for which any one of these circles is described as a central orbit.*

4. CIRCULAR ORBITS THROUGH A POINT. *Theorem I. Through any point* (u_1, θ_1) *there passes in each direction one and only one circle of the family* (5). For, if the given direction makes with the radius vector to that point an angle ψ_1, then, since $\tan \psi = r \, d\theta/dr = -u \, d\theta/du$, equation (3) gives for any circle of (5) on replacing a by A and $a^2 - d^2$ by n^2:

(6)
$$u_1{}^2 \csc^2 \psi_1 = 4A^2 u_1{}^4 [n^4 u_1{}^2 + 1]^{-2}.$$

This equation determines a single real positive value of A, which is never less than n, since $\csc^2 \psi_1 \geq 1$ and $n^2 u_1{}^2 + 1 \geq 2nu_1$:

(7)
$$A^2 = \csc^2 \psi_1 \cdot (n^2 u_1{}^2 + 1)^2 / 4u_1{}^2 \geq n^2.$$

Hence $D^2 = A^2 - n^2$ gives a real value for D. The substitution of the latter value, with the coordinates of (u_1, θ_1) in (5) determines K uniquely (except for additive multiples of 2π) and real.

Theorem II. There is a unique velocity with which a particle projected at any point in any direction will describe a circle. For, by the preceding theorem, A is determined uniquely; and hence $HA = m$ determines H uniquely. And since $H = v_1 r_1 \sin \psi_1$ in any central orbit, there is but one possible value for v_1. Now to describe the required circle with the determined value of H, the particle would have to be attracted by a force whose law is precisely (4); hence, conversely, projection in the given direction with the velocity v_1 will result, under the law (4), in a circular orbit.

5. PROPERTIES OF THE FAMILY OF ORBITS. From the constancy of AH it follows that the constant of areas required in any circular orbit decreases as the radius of the circles increases. From the constancy of $A^2 - D^2$ it follows that the radius of the circle must always increase with the distance of its center from the pole. There is no upper limit for A or D, but there is a lower limit (not zero) for A: viz., the smallest possible circular orbit is that one whose center lies at the center of force. But the two invariants have more exact interpretations, which are both simple and interesting.

---

*In a paper "On the Law of Gravitation in the Binary Systems," [*American Journal of Mathematics.* vol. 31, pp. 62–85], the writer discussed the family of conics by a law of which Type I is a limiting case. The present family of circles, however, is not included among those conics; for each law there considered admitted only one circular orbit, and the center was at the pole. Certain questions raised and answered below would not have arisen in connection with the family of conics.

Property 1. The curvature at either apse is proportional to the constant of areas required for the orbit. For the curvature is $1/A$ which equals H/m.

Property II. The product of the two apsidal distances is constant. For these distances are $A-D$ and $A+D$ whose product equals n^2.

Property III. All circular orbits whose apsidal radii vectores are collinear pass through two fixed points on the perpendicular drawn to the line of centers at the center of force. For the half-chord intercepted on this perpendicular, being a mean proportional between $A-D$ and $A+D$, is constant and equal to n

Another (and more useful) statement of this property is that *the vectorial angle from either apse to the radius vector of length n is constant and equal to $\pi/2$.*

§ 3. OTHER LAWS ADMITTING PROPERTIES I — III.

6. ADMITTING PROPERTY I. Two of these three properties were found also in the family of conics mentioned above; and it is natural to inquire whether there are still other laws of force whose trajectories will possess such properties. Attention will be confined to those cases where the force is a continuous function of the distance.

Let the force be denoted by $u^2P(u)$, and let $F(u)=2\int P(u)du$. [Since the continuity of $P(u)$ follows from the hypothesis that f is a continuous function of r, $P(u)$ is integrable; thus $F(u)$ exists and is continuous.]

A first observation is that any such force, whether attractive or repulsive, admits families of orbits having pericenters at arbitrary distances; and, if everywhere attractive, admits also families having apocenters at arbitrary distances. For a particle may be so projected at any distance $1/\beta$ that the initial direction is perpendicular to the initial radius vector, so that $du/d\theta=0$ for $u=\beta$; and the initial velocity may be such as to give h any desired value. Moreover the standard differential equation of central orbits,

(8) $$f=h^2u^2(u+d^2u/d\theta^2)=u^2P(u),$$

shows that $d^2u/d\theta^2 \gtrless 0$ at $u=\beta$ according as $h^2 \gtrless P(\beta)/\beta$. Hence whatever the sign of $P(\beta)$, h may be chosen so large that $d^2u/d\theta^2<0$, making u a maximum at the initial point. And, if $P(\beta)>0$ [force attractive], it is also possible to choose h so small that u is a minimum at $u=\beta$.

Secondly, since the expression for the curvature [see No. 2] becomes at an apse $u+d^2u/d\theta^2$, or simply $P(u)/h^2$, Property I would require: $h=\lambda P(\beta)/h^2$, or

(9) $$h^3=\lambda.P(\beta),$$

where λ is a constant factor of proportionality. Thus for *any* value of λ, h is defined as a function of β so as to insure Property I. It remains to be shown simply that λ may have such values that h, defined by (9), will meet the condition above for the existence of a pericenter or apocenter at $u = \beta$. Now to have $h^2 > P(\beta)/\beta$ is to have $[\lambda P(\beta)]^2 > [P(\beta)/\beta]^3$, or $\lambda^2 > P(\beta)/\beta^3$. But for any interval of β, $0 < A \lesseqgtr \beta \lesseqgtr B$, $P(\beta)$ being continuous has an upper bound M. Hence, if λ^2 be chosen greater than M/A^3, the condition is satisfied. And if $P(\beta) > 0$ always, the reverse inequality may be treated in like manner.

Therefore, *every central force which is a continuous function of the distance admits families of orbits having pericenters and possessing Property I; and, if everywhere attractive, families of orbits with apocenters, having that property.*

Forces admitting orbits with both a pericenter and an apocenter are somewhat special as will appear in the next paragraph. For the orbits to have Property I at their pericenters only, any continuous force whose trajectories have two apses is admissible, the corresponding apocenters being determined as follows: One integration of (8) gives

$$(10) \qquad\qquad h^2(u^2 + (du/d\theta)^2) = c + F(u),$$

where c is an arbitrary constant. If β and α denote, respectively, the pericentral and apocentral values of u, then

$$(11) \qquad\qquad h^2(\beta^2 - \alpha^2) = F(\beta) - F(\alpha).$$

Now if β be allowed to vary, and h be determined by (9), then (11) will determine α in those cases where the orbit has two apses. If, however, Property I is to hold for both apses, there is a further restriction upon the force, given by $\lambda P(\beta) = \mu P(\alpha)$, μ being the factor of proportionality for apocenters. A case of this sort is treated later [in No. 9].

7. ADMITTING PROPERTY II. If an orbit has two apsidal distances, $1/\alpha$ and $1/\beta$, Property II requires simply that $\alpha\beta = k^2$, a constant. As remarked, however, there is a second category of forces none of whose trajectories have two apsidal distances; an example is the case where the force varies inversely as the cube of the distance. Still other laws admit trajectories consisting of two branches separated by a region of imaginary motion, —one branch having only a pericenter and extending to infinity, the other having only an apocenter and reaching to the center of force. In such a case which branch is actually described in the motion of the particle depends upon the initial distance of the particle; and the path really has but one apse. It is easy to find conditions upon the force, which are necessary and sufficient to place it in any of the three categories.

Theorem I. In order that a force, f, shall admit orbits extending from one apse to another, it is necessary and sufficient that f/u^3 be a decreasing function of u throughout some interval of u, say $[a, b]$. For if there is an apocenter at $u=a$ and a pericenter at $u=\beta$ $[\beta>a]$, the necessary constant of areas is given by (11); and moreover it is necessary that $h^2 d^2 u/d\theta^2$, or $-h^2 u+P(u)$, be positive for $u=a$ and negative for $u=\beta$; thus

$$(12) \qquad\qquad P(a)/a>h^2>P(\beta)/\beta.$$

If $P(u)/u$ is nowhere a decreasing function of u, it is clear that the inequality (12) is impossible; thus the hypothesis is necessary.

It is also sufficient; for let $a=a$ and $\beta=b$. Then the use of the generalized theorem of mean value in (11) gives the constant of areas for which $du/d\theta$ vanishes at $u=a$ and at $u=\beta$:

$$(13) \qquad\qquad h^2=\frac{F(\beta)-F(a)}{\beta^2-a^2}=\frac{P(u_1)}{u_1},$$

where $a<u_1<\beta$, the theorem being valid here, since $F(u)$ and u^2 together with their derivatives are continuous functions in $[a, \beta]$, and the latter dedominator does not vanish in the interval. Now by hypothesis,

$$(14) \qquad\qquad \frac{P(a)}{a} > \frac{P(u_1)}{u_1} > \frac{P(\beta)}{\beta} ;$$

so that (12) is satisfied, and there is an apocenter where $u=a$ and a pericenter where $u=\beta$. Whether the orbit extends from $u=a$ to $u=\beta$ or whether there are still other apses between these values of u (for the same values of h) does not affect the theorem. For consider the next apsidal value above $u=a$, say $u=a'$. Then since $h^2=P(u_2)/u_2$ where $a<u_2<a'$; and since this shows that $h^2>P(a')/a'$, the apse at $u=a'$ is a pericenter. A similar argument shows that the apse nearest to $u=\beta$, say $u=\beta'<\beta$, would have to be an apocenter. In any event the same branch has both a pericenter and an apocenter.

Corollary 1. If f/u^3 decreases everywhere, then any two distances can be the apsidal distances in an orbit. For the constant of areas required for the vanishing of $du/d\theta$ would be such as to give a pericenter at the lesser distance and an apocenter at the greater.

Theorem II. In order that a force, f, admit trajectories consisting of two branches, one having an apocenter only, and one a pericenter only, it is sufficient that f/u^3 be everywhere an increasing function of u, and necessary that f/u^3 be an increasing function in some interval $[a, b]$. For the existence of a pericenter at $u=\beta$ and an apocenter at $u=a$ $[a>\beta]$ requires the

inequality (12), which can be satisfied only if f/u^3 is an increasing function in some interval between β and α. Further, since (14) is satisfied if f/u^3 is everywhere an increasing function, the apses at $u=\alpha$ and $u=\beta$ must be, respectively, an apocenter and a pericenter; and there are no further apses by Theorem I, the necessary condition not being fulfilled.

Corollary 2. If f/u^3 *is constant* [the case of the inverse cube of the distance], *only one apse is possible in a trajectory.* For the conditions necessary for two apses either in the same branch or in different branches are not satisfied.

Corollary 3. *There is no apsidal distance between* $1/\alpha$ *and* $1/\beta$ *in the orbits of Theorem I.* For between the apocentral value $u=\beta'$ and the pericentral value $u=\alpha'$, f/u^3 would have to be somewhere an increasing function.

Theorem III. *Any law admitting trajectories with two apsidal distances admits families having Property II.*

Case I. When there is an interval $[a, b]$ throughout which f/u^3 decreases, any two apsidal values α and β $[a \leqq \alpha < \beta \leqq b]$ determine a value of h for which the orbit lies in the region $u=a$ to $u=b$ and has two apses. Let $ab = k^2$, and choose arbitrarily $k \geqq \alpha \geqq a$, and $\beta = k^2/\alpha$. Then the orbit lies between $u=\alpha$ and $u=\beta$, and has Property II.

Case II. When f/u^3 always increases with u, select k and α arbitrarily $[\alpha > k]$, and let $\beta = k^2/\alpha$. Then the vanishing of $du/d\theta$ for $u=\alpha$, β, determines a value of h, $h^2 = P(u_1)/u_1$, such that $P(\alpha)/\alpha < h^2 < P(\beta)/\beta$. Thus the trajectory has two branches of the kind considered in Theorem II. By varying α a family is obtained having Property II.

8. ADMITTING PROPERTY III. Not all forces admit orbits in which the angle from the radius vector of a given length to the nearest pericenter is $\pi/2$. For example, the laws represented by $f=kr^n$ where $n>1$ have their apsidal angles all less than $\pi/2$.* But for any force, f, such that $u.f$ is an increasing function of u, all apsidal angles are greater than $\pi/2$; and by Corollary 1, if also f/u^3 is everywhere a decreasing function, any pair of apsidal values α and β are admissible.

Let c be an arbitrarily chosen constant; and let θ denote the angle from the radius vector where $u=c$ to the nearest pericentral line, where $u=\beta>c$. Then

$$(15) \qquad \theta = \int_c^\beta \frac{d u}{\sqrt{\{\beta^2 - u^2 + [F(u) - F(\beta)]/h^2\}}}.$$

Now for sufficiently great values of h, this integral differs arbitrarily little from

---

*This and the following statement are established by a test contained in an unpublished paper by the writer, presented to the American Mathematical Society, December 30, 1908.

(16) $$\int_\beta^c du/\sqrt{(\beta^2-u^2)}, \text{ or } \pi/2-\arcsin(c/\beta);$$

hence there are values of h large enough to make $\theta<\pi/2$. But if a be chosen sufficiently near c [$a<c$], $\Theta-\theta$ may be made arbitrarily small [Θ denoting the apsidal angle]; so that, since $\Theta>\pi/2$, h can be so chosen that $\theta>\pi/2$. Between this value of h and the "sufficiently great values," there must exist a value for which $\theta=\pi/2$.

By varying β a family of orbits is obtained having Property III. For any value of β the necessary value of h is given by (15) on placing $\theta=\pi/2$; and a is given by (11) when β and h are known.

There is likewise a family having Property III with respect to the apocenters; for with a given a, h can be selected so small that θ shall be arbitrarily small; or β selected so near c that θ shall be greater than $\pi/2$. Hence, for a certain value of h, $\theta=\pi/2$.

The determination of h in each case is unique; for in (15) if h be assigned two different values h_1 and h_2, $h_2>h_1$, the integrand involving h_2 is smaller throughout than that involving h_1, so that only for one of these values can $\theta=\pi/2$.

9. ADMITTING PROPERTIES I AND II. The preceding sections show that no one of the three properties serves to characterize any particular law of force. But it is quite otherwise if laws be sought admitting both Properties I and II.

For if the constant of areas be proportional to the curvature at each apse; then $h^3=\lambda P(\beta)=\mu P(a)$, where λ and μ are constants. And if the family is to have $a\beta=k^2$, λ and μ must be equal, since as a and β each approach the value k, h^3 approaches $\lambda P(k)$ or $\mu P(k)$. Therefore, also $P(k^2/\beta)=P(\beta)$; and from (11) follows

(17) $$F(\beta)-F(k^2/\beta)=(\beta^2-k^4/\beta^2)[\lambda.P(\beta)]^{\frac{2}{3}},$$

which must hold as β varies through the range of values taken in the family. Differentiation of (17) with respect to β gives on replacing $P(k^2/\beta)$ by $P(\beta)$:

(18) $$(1+k^2/\beta^2)P(\beta)=(\beta+k^4/\beta^3)[\lambda P(\beta)]^{\frac{2}{3}}$$
$$+(\lambda/3)(\beta^2-k^4/\beta^2)[\lambda P(\beta)]^{-\frac{1}{3}}P'(\beta),$$

or substituting kx for β and $\phi(x)$ for $\lambda.P(kx)$:

(19) $$\frac{d\phi}{dx}+3\frac{x^4+1}{x^5-x}\phi=\frac{3}{\lambda k}\cdot\frac{\phi^{\frac{4}{3}}}{x^2-1},$$

a differential equation which is of Bernoulli's type, and is reducible to a lin-

ear equation by the substitution $\phi=y^{-3}$. A particular integral is $y=(x^2+1)$ $/2\lambda kx$, while the solution of the auxiliary equation is $y=c\sqrt{(x^4-1)}/2\lambda kx$; so that the general solution is

(20) $$\phi^{-\frac{1}{3}}=y=[c\sqrt{(x^4-1)}+x^2+1]/2\lambda kx,$$

where c is an arbitrary constant; or

(21) $$P(\beta)=8k^6\lambda^2\beta^3[c\sqrt{(\beta^4-k^4)}+\beta^2+k^2]^{-3}.$$

Hence the most general law of force, $f=u^2P(u)$, would be

(22) $$f=8k^6\lambda^2\cdot\frac{u^5}{[c\sqrt{(u^4-k^4)}+u^2+k^2]^3}.$$

But, whether c is real or imaginary $[c\neq0]$, (22) gives imaginary values for f either when $u>k$ or else when $u<k$; and such a force would not admit real orbits in both parts of the plane. Thus the only admissible value of c is zero; and (22) then reduces to the same form as (4).

      Hence *laws of Type I are the most general which admit families of orbits possessing Properties I and II;* and these two properties serve, therefore, to characterize laws admitting circular orbits whose centers are elsewhere than at the center of force.

§ 4. The Families for Particular Laws.

      10. For Newton's Law. Every orbit described under a force operating according to Newton's law is a conic having a facus at the center of force. Expressed in terms of the constants of the conic, the force is given by

(23) $$f=h^2u^2/a(1-e^2),$$

so that, for all the orbits, $h^2/a(1-e^2)$ must be constant. If also h varies as the curvature at any apse, $1/a(1-e^2)$ or a/b^2, it follows that $a(1-e^2)$, and hence also h, must be constant in the family. Hence, for the Newtonian law, the family having Property I is the *totality of conics* having a certain curvature at a vertex, or what is equivalent, *having latera recta of a certain length.* Evidently there is an infinitude of such families, obtained by changing the arbitrary constant. And further, each of these families possesses Property III also.

      Again, to have the product of the apsidal distances constant requires the constancy of $a^2(1-e^2)$, or of the minor axis. Hence for Newton's law,

the family possessing Property II is the set of conics having minor axes of an arbitrary constant length.

11. FOR THE LAW OF THE DIRECT DISTANCE. All orbits described under a force varying directly as the distance, are conics whose centers lie at the center of force. The force, expressed in terms of the constants of the conics, is given by

$$(24) \qquad\qquad f = h^2/a^2b^2u;$$

so that h^2/a^2b^2 must be constant for all the orbits. If also h is proportional to a/b^2, it is necessary that b^6, and hence also b, remain invariant. Thus, for this law, the family possessing Property I is the set of conics having minor axes of a given length. The constant of areas must vary as the major axis in this family. To have Property I at the pericenters, interchange b and a in the preceding statements. Each of these families has Property III also.

Finally, to have the product of the apsidal distances constant requires the constancy of $a.b$ and also of h. Hence, for the law of the direct distance, the family possessing Property II is the set of conics having the product of the axes constant, or what is equivalent for ellipses, having a given (arbitrary) area.

ON THE IRREDUCIBILITY OF CERTAIN POLYNOMIALS.

By JACOB WESTLUND, Purdue University.

The object of the following note is to determine whether the two polynomials

$$f_1(x) = (x-a_1)(x-a_2) \ldots (x-a_n)-1 \text{ and}$$
$$f_2(x) = (x-a_1)(x-a_2) \ldots (x-a_n)+1,$$

where $a_1, a_2, \ldots, a_n$ are distinct integers, are reducible or irreducible.

Let us first consider $f_1(x)$. If $f_1(x)$ were reducible we would have

$$f_1(x) = \phi(x)\psi(x),$$

where $\phi(x)$ is irreducible and of a lower degree than n. Then since

$$\phi(a_i)\psi(a_i) = -1, \ i = 1, 2, \ldots, n,$$

we must have